Unemployment Dispersion and City Configurations: Beyond the Bid Rent Theory

Vincent Boitier

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Abstract

In the present article, I provide a simple urban theory where agents do not bid for land. In absence of this baseline mechanism, I show that the spatial allocation of agents is governed by a Nash equilibrium. I underline the role of asymmetric local congestion effects in insuring the existence and the uniqueness of such an equilibrium. I then use this new framework to account for spatial variation in unemployment within big cities. Namely, applying this setting in an urban search model, I demonstrate that the obtained framework can generate a large number of new city configurations in which the local unemployment rate behaves differently. I also determine conditions for which each configuration may appear. I finally prove, the existence and the uniqueness of a labor market equilibrium for each urban pattern and I draw a link between the latter and the allocation of workers throughout space.

JEL Classification: J63, J64, R14.

Keywords: Matching models, Local congestion effects, Unemployment dispersion, Segregation, Bid rent theory.

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1 Introduction

The bid-rent theory, determining how land prices and housing demand vary across space, has become the cornerstone in urban economics. In a world where distinct groups of individuals have an intrinsic use of land, this theory states that space is allocated to the use that bids the most for it. Accordingly, land use in a given location generally is of one type. Land use can be of several types if, and only if, at least, two populations share the same bid-rent function. This particular case can never occur (see, for example, Mossay and Picard (2013)) or can prevail under a very specific parameterization (see Fujita and Ogawa (1980, 1982) and Lucas and Rossi-Hansberg (2002)). This also involves that segregation, a state in which agents get separated into different neighborhoods, commonly arises as an outcome. Other configurations as, for instance, integrated situations where different populations of individuals share the same places of residence, rarely emerge.

Though it is fair to observe that cities show spatial sorting of inhabitants, other empirical facts totally debunk the existence of cities organized according to concentric rings of land use. For instance, it has been well established that the location of employed and unemployed workers are strongly inter-dispersed (see, amonger others, Wheeler (1998), Topa (2001, Figures 1-3), Gobillon and Selod (2007, Carte 1), Dujardin, Selod and Thomas (2008, Figure 2), Dujardin and Gofette-Nagot (2010, Figure 1)). Namely, spatial data display a large amount of heterogeneity in terms of unemployment dispersion and document that segregation is the exception rather than the rule. Unfortunately, theoretical literature has been unable to account for this spatial intra-variation in the unemployment rate (see Zenou (2009a)). The latter mostly pays attention to segregated cities where there are only two local unemployment rates: either 0 % or 100 % (see, within a large literature, Wasmer and Zenou (2002), Smith and Zenou (2003), Kawata and Sato (2012), Xiao (2013)).[1] This is explained by the fact that the models of this literature suppose that land is driven by bid-rent principle which, as previously mentioned, prevents the existence of cases where unemployed and employed workers live together.

The present article aims at accounting for spatial dispersion in the unemployment rate within cities by providing an alternative urban model where agents do not bid for land. My theoretical point is to highlight that, in the absence of this standard mechanism, the spatial distribution of workers is driven by a Nash equilibrium and to show the relative importance of asymmetric local congestion effects in determining the existence and the uniqueness of such an equilibrium. Moreover, my objective is also to study the new city configurations that may emerge within this framework. Therefore, the purpose of this article is qualitative

[1] Only Wasmer and Zenou (2006) find a configuration where two segregated areas of the unemployed surround a zone where the unemployed and the employed co-exist.

in nature and not quantitative. Thence, I operate in two steps.

First, I study the main characteristics of a standard urban model but where agents do not bid for space. To this end, I use Cirant (2014) and I extend the framework of Cardaliaguet (2012) to several populations of agents. More precisely, I develop a game with infinite number of agents sharing a common set of strategies and being distributed (for the sake of simplicity) in two different groups. Each player, whatever their group, has to choose a unique strategy maximizing a payoff function. The key feature of this game is that the utility functions depend on the selected strategy and on the density of other agents playing this strategy. I then carry out the study (focusing on existence and uniqueness) of a Nash equilibrium when the number of individuals becomes infinite.

The central result of this part is to demonstrate that a unique Nash equilibrium exists if utility functions exhibit asymmetric congestion (or competition) effects. In other words, existence and uniqueness hinge on two conditions. On the one hand, players' payoff functions have to decrease with the density of other players. Such preferences express local congestion effects. On the other hand, these negative local externalities must be not the same for both populations of players.

Second, I use this methodology to build and examine the properties of a simple urban model with endogenous unemployment. I specifically consider a monocentric city with a continuum of locations filled with an infinite number of firms, absent landlords and workers. Search in the city is random. Firms are exogenously located in the city center, do not consume any space and compensate workers for their spatial costs (see Zenou (2009b)). Absent landlords own houses that are exogenously supplied by a competitive market. Workers have hyperbolic preferences as in Mossay and Picard (2011, 2013) and as in Blanchet et al. (2012). They choose the location and size of their dwellings knowing that they would face high relocation costs (see Zenou (2003, 2006, 2009b)). They can also remain in two different states : either employed or unemployed. When they are employed, they commute daily to the city center in order to work. When they are unemployed, they occasionally commute to the center to look for a job. Another difference between workers is that employed workers incur an additional cost to live with other employed workers. This asymmetric local congestion effect can be interpreted as a congestion effect in road or a competition effect in the consumption of neighborhood goods.

In this setup, workers face a trade-off. They have an incentive to live close to the city center to avoid paying high transport costs, but they also anticipate that these locations are precisely those to which a high number of other inhabitants will reside in. As a consequence, they also are encouraged to live in remote places of residence to escape congestion.

This induces that workers endogenously select a place of residence according to their spatial preferences and the strategy of other worker who are captured by the residential density of the unemployed and employed. Consequently, the allocation of workers in space is no longer related to the bid-rent theory but with the Nash equilibrium described above. It determines an endogenous spatial distribution of unemployed and employed workers and so an endogenous unemployment dispersion. Last, equivalently to Zenou (2009a) , wages, job creation and unemployment are determined according to the labor market. The latter is modeled by a search and matching equilibrium (i.e. Nash bargaining, job creation equation and Beveridge curve).

Within this environment, I show that many new urban configurations can appear. The city can be: purely integrated if both groups of workers live together and in the same proportions; segregated if populations of workers get separated into two different parts of the city; integrated when employed and unemployed workers share the same places of residence but in different proportions; incompletely and purely integrated if the city is purely integrated into a part of space, whereas in other zones, the city is segregated; incompletely integrated if the city is integrated into a zone of the city while in other areas, it is segregated. Among incompletely and purely integrated cities, two sub-configurations can be found: a central core of mixed workers surrounded by a segregated part of unemployed or employed workers. Likewise, among incompletely integrated cities, four sub-patterns can be pointed out: a central core of employed workers surrounded by a peripheral integrated ring of workers, a central core of mixed-ring workers surrounded by a peripheral segregated part of employed or unemployed workers and both a central core and a peripheral ring of segregated areas separated by an intermediate ring of mixed workers. In each configuration and sub-configuration, the behavior of the local unemployment rate is different. In purely integrated cities, it is uniformly distributed. In segregated cities, it is degenerated (i.e. 0 % or 100 %). In integrated cities, it is continuously but non-uniformly distributed. In incompletely and purely integrated cities, it is uniformly distributed in the core of the space and then degenerated in the fringes of the city. In incompletely integrated cities, it is continuously distributed in some areas and degenerated in other parts of the city.

I then emphasize very simple and analytical conditions for which each urban situation can emerge as a balance between agglomeration and dispersion forces. By studying these conditions, I demonstrate that the predominance of one of these patterns relies on the relative importance of the employment rate, the worker's search effort, the worker's bargaining power, congestion between employed workers, preference for land and transport costs.

To conclude, for each urban configuration, I prove the existence and the uniqueness of a labor market equilibrium (i.e. an equilibrium labor market tightness and an equilibrium unemployment rate). I also characterize the link between the outcome of this labor market equilibrium and the spatial distributions of workers. For example, in the present model, the intra-variation of the local unemployment rate does not impact the global unemployment rate of the city. Only the spatial concentration of employed workers plays a significant role. A direct consequence of this fact is that segregated cities always show lower unemployment rates than integrated ones.

The article is organized as follows. Section 2 describes the theoretical game where the role of local congestion effects is underlined. Section 3 presents the urban search model. Section 4 concludes.

2 Population Games, Nash Equilibrium and Local Congestion Effects

This article is based on one particular form of Nash equilibrium. As the latter is implicitly presented but not studied in urban economics, I briefly highlight some useful definitions and results using Cirant (2014). I notably describe a population game where individuals are engaged with intra- and inter-group local interactions. I analyze the characteristics (existence and uniqueness) of the Nash equilibrium of such a game when the number of agents becomes very large. Technically speaking, this game consists in extending the framework of Cardaliaguet (2012, Section 2) to several populations of players. All material developed in this section is also related to routing games (see Haurie and Marcotte (1985)), large crowding games (see Milchtaich (2000)) and mean field games (see Lasry and Lions (2007)).

2.1 Setup

Let $\mathcal{X}$ be a compact subset of $\mathbb{R}$ and $\mathcal{M}(\mathcal{X})$ be the set of absolutely continuous Borel probability measures on $\mathcal{X}$ (with respect to the Lebersques measure) denoted by μ and having density also denoted by μ in $C^0(\mathcal{X})$ with $C^0(\mathcal{X})$ the set of continuous functions. This set is endowed with the Kantorowich-Rubinstein distance:

$$d(\mu,\tilde{\mu}) = sup\left\{\int_{\mathcal{X}} g(x)d(\mu-\tilde{\mu})(x) : g \in C^{0,1}(\mathcal{X}), Lip(g) \leq 1\right\} \tag{1}$$

where $C^{0,1}(\mathcal{X})$ is the set of continuous and differentiable functions on $\mathcal{X}$ and with $Lip(g)$ the minimal Lipschitz constant for g. This implies that d metricizes the weak-* convergence on $\mathcal{M}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ is compact for d. Subsequently, I consider a continuum of agents distributed into two different groups index $k \in \{1,2\}$.[2] In a given population k, players are

[2] A generalization for $k \in \mathbb{N}_+^*$ is possible without any difficulty.

homogenous players and have to choose a unique action from $\mathcal{X}$ a common set of actions. In order to do so, they maximize a continuous payoff function $\mathcal{A}_k : \mathcal{X} \times (0, \infty)^2 \to \mathbb{R}$ that depends on x their strategies and $\mu_k(x)$ the density of individuals (in each population) playing the same strategy:

$$\mathcal{A}_k \left(x_k^i, \mu_1(x_k^i), \mu_2(x_k^i)\right) \tag{2}$$

with x_k^i the strategy of player i that belongs to group k. For example, one could consider the following linear case:

$$\begin{cases} \mathcal{A}_1 \left(x, \mu_1(x), \mu_2(x)\right) = a - tx - \phi_{11}\mu_1(x) - \phi_{12}\mu_2(x) \\ \mathcal{A}_2 \left(x, \mu_1(x), \mu_2(x)\right) = a - tx - \phi_{21}\mu_1(x) - \phi_{22}\mu_2(x) \end{cases} \tag{3}$$

with $a, t, \phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$. Finally, note that, if $\mathcal{A}_k$ decreases with the proportion of other individuals, agents are encouraged to play differently. Such preferences reflect what is commonly called local congestion effects.

2.2 Equilibrium: Definition, Existence and Uniqueness

A Nash equilibrium, in the sense that all agents whatever their groups play best response, is given by:[3]

Definition 1 *A vector $(\mu_1^*, \mu_2^*) \in \mathcal{M}(\mathcal{X})^2$ is a Nash equilibrium if, and only if:*

$$\int_{\mathcal{X}} \mathcal{A}_k \left(x, \mu_1^*(x), \mu_2^*(x)\right) d\mu_k^*(x) = \sup_{\mu \in \mathcal{M}(\mathcal{X})} \int_{\mathcal{X}} \mathcal{A}_k \left(x, \mu_1^*(x), \mu_2^*(x)\right) d\mu(x) \tag{4}$$

In line with the game theory literature, an equilibrium is a state where the payoff function $\mathcal{A}_k$ is maximized if the density of players in population k is non-zero. Put differently, the equilibrium can be re-written as:

$$Supp(\mu_k^*) \subseteq arg \max_{x \in \mathcal{X}} \mathcal{A}_k \left(x, \mu_1^*(x), \mu_2^*(x)\right) \tag{5}$$

For the sake of simplicity, I will use this standard formulation (see, among others, Sandholm (2001)) to characterize the spatial equilibrium of the urban search model. Now, let me tackle the question of the existence and uniqueness of this equilibrium. More precisely, I get:

Proposition 1 *There exists at least one vector $(\mu_1^*, \mu_2^*) \in \mathcal{M}(\mathcal{X})^2$ satisfying (4).*

[3] This Nash equilibrium can be found by taking the limit of a static game (see Cardaliaguet (2012, Theorem 2.4 and Theorem 2.7). However, it is also possible to derive it from other frameworks. For example, equilibrium (3) can be viewed as a stationary mean field game: see Cirant (2014) or take Theorem 2.8 and see Section 2.7 in Lasry and Lions (2007). It is also related to models of interacting agents: take equation (16) in Lemoy, Bertin and Jensen (2011) and set $T \to 0$.

Additionally, uniqueness occurs under the following assumptions:

Proposition 2 *Suppose that:*

$$\int_{\mathcal{X}} \sum_{k=1}^{2} \mathcal{A}_k \left(x, \mu_1^*(x), \mu_2^*(x)\right) - \mathcal{A}_k \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d\left(\mu_1^* - \tilde{\mu}_1\right)(x) < 0 \tag{6}$$

for all $\mu_1^* = \tilde{\mu}_1$ *and* $\mu_2^* = \tilde{\mu}_2$*, then, there is at most a vector* $(\mu_1^*, \mu_2^*) \in \mathcal{M}(\mathcal{X})^2$ *satisfying (4).*

This Proposition is in line with Lasry and Lions (2007) and identical to the one of Cirant (2014, Theorem 5.1). Moreover, using Lagrange Theorem and assuming that $\mathcal{A}_1$ and $\mathcal{A}_2$ are differentiable, it appears that a sufficient condition for (6) is that:

$$A(x, \mu_1, \mu_2) + A(x, \mu_1, \mu_2)^T \text{is negative semi-definite} \tag{7}$$

with:

$$A(x, \mu_1, \mu_2) = \begin{pmatrix} \frac{\partial \mathcal{A}_1(x,\mu_1(x),\mu_2(x))}{\partial \mu_1(x)} & \frac{\partial \mathcal{A}_1(x,\mu_1(x),\mu_2(x))}{\partial \mu_2(x)} \\ \frac{\partial \mathcal{A}_2(x,\mu_1(x),\mu_2(x))}{\partial \mu_1(x)} & \frac{\partial \mathcal{A}_2(x,\mu_1(x),\mu_2(x))}{\partial \mu_2(x)} \end{pmatrix} \tag{8}$$

and $A(x, \mu_1, \mu_2)^T$ the transpose of the matrix $A(x, \mu_1, \mu_2)$. For example, if I consider the linear case (3), this sufficient condition holds if: $\phi_{11} = \phi_{21} = \phi_{22} = -\phi$ and $\phi_{12} = -\phi - \varphi$ with $\phi, \varphi > 0$ two constants. However, notice that the condicient is not verified for the symmetric case: $\phi_{11} = \phi_{12} = \phi_{21} = \phi_{22} = -\phi$. Thus and inuitively, uniqueness emerges if two conditions are completed:

1. The payoff functions are negatively correlated with the density of other players: $\frac{\partial \mathcal{A}_k(x,\mu_1(x),\mu_2(x))}{\partial \mu_k(x)} < 0$. This second condition asserts that uniqueness prevails if utility functions exhibit local congestion effects.

2. The matrix A is not symmetric. This indicates that local interactions between individuals are asymmetric: $\frac{\partial \mathcal{A}_j(x,\mu_1(x),\mu_2(x))}{\partial \mu_k(x)} \neq \frac{\partial \mathcal{A}_j(x,\mu_1(x),\mu_2(x))}{\partial \mu_k(x)}$. Equivalently, multiplicity appears when players are symetric in terms of their loss from intra- and inter-groups interactions: $\frac{\partial \mathcal{A}_j(x,\mu_1(x),\mu_2(x))}{\partial \mu_k(x)} = \frac{\partial \mathcal{A}_j(x,\mu_1(x),\mu_2(x))}{\partial \mu_k(x)}$.

I will use this methodology in an urban search model.[4] More accurately, I will show that the standard urban search model, but only where space is not driven by the bid-rent theory, shares identical elements with the previously framework. Hence, the appropriate equilibrium will be the one expressed in equation (4) and existence and uniqueness of such equilibrium will be provided by Proposition 1 and Proposition 2.

[4]This setting can also be applied to many other economic fields: macroeconomics, labor economics, Schelling models... See Boitier and Vatan (2014) for a example in international trade.

2.3 Link with Urban Economics

In this subsection, I show that the previous game corresponds to the standard urban land use model but where agents do not bid for land. For this purpose, let $\mathcal{X}$ denote a featureless space that hosts two populations (indexed k) of workers. In a given population k, there is a unit mass of homogenous workers associated with the following continuous function:

$$\mathcal{Z}_k(\sigma_k(x), \zeta_k(x)) \tag{9}$$

and the following budget constraint:

$$\sigma_k(x) + R(x)\zeta_k(x) = I_k(x) = y_k - t(x) \tag{10}$$

with $\sigma_k(x)$ the amount of composite consumer good and $\zeta_k(x)$ the size of houses, $R(x)$ the rent per unit of land, y_k a fixed exogenous income, $t(x)$ the transport costs and $I_k(x)$ the net revenue. In this mere environment, they make two decisions so that:

$$\max_{x,\zeta_k(x)} \{\mathcal{Z}_k(I_k(x) - R(x)\zeta_k(x), \zeta_k(x))\} \tag{11}$$

I solve this problem in two steps. In a first step and for x fixed, maximizing program (11) with respect to $\zeta_k(x)$, I obtain the classical Marshallian demand function:

$$\zeta_k^*(x) = h^*\left(I_k(x), R(x)\right) \tag{12}$$

and the (continuous) indirect utility function:

$$\mathcal{A}_k(I_k(x), R(x)) = \mathcal{Z}_k(I_k(x) - R(x)\zeta_k^*(x), \zeta_k^*(x)) \tag{13}$$

with h^* a continuous function that increases with $I_i(x)$ and decreases with $R(x)$ (i.e. land is a normal good). In a second step, for $\zeta_i^*(x)$ given and noting that the land market clears:[5]

$$\zeta_1^*(x)\mu_1(x) + \zeta_2^*(x)\mu_2(x) = 1 \tag{14}$$

the rent per unit of land can be re-written as:

$$R^*(x) = g^*\left(I_1(x), I_2(x), \mu_1(x), \mu_2(x)\right) \tag{15}$$

where g^* is a continuous function. In addition, combining equation (13) and equation (15), I get:

$$\max_x \mathcal{A}_k(x, \mu_1(x), \mu_2(x)) = \max_x \{\mathcal{Z}_k(I_k(x) - R^*(x)\zeta_k^*(x), \zeta_k^*(x))\} \tag{16}$$

Thence, the suited equilibrium associated with problem (16) is the above Nash equilibrium and if $\mathcal{A}_1$ and $\mathcal{A}_2$ comply with conditions in Proposition 2, this equilibrium is unique.

[5] I suppose that land intensity equals to 1. This does not determine the nature of the results.

Last but not least, notice that the standard urban model only makes an extra hypothesis. Namely, it is assumed that:

$$R^*(x) = max\left\{\Psi(\mathcal{A}_1^*, x), \Psi(\mathcal{A}_2^*, x), R_A\right\} \tag{17}$$

with R_A the agricultural rent and so that

$$\Psi(\mathcal{A}_k^*, x) = \max_{\sigma_k(x),\zeta_k(x)} \left\{ \frac{y_k - t(x) - \sigma_k(x)}{\zeta_k(x)} | \mathcal{Z}_k(\sigma_k(x), \zeta_k(x)) = \mathcal{A}_k^* \in \mathbb{R} \right\} \tag{18}$$

where $\Psi(\mathcal{A}_k^*, x)$ is the bid-rent function of workers belonging to population k and $\mathcal{A}_k^*$ the equilibrium utility for population k.

3 Urban Search Model without the Bid Rent Theory

The model considered hereafter uses the methodology outlined in Section 2. Although the latter is based on general functions, in what follows, I will pin down linear functional forms for $\mathcal{A}_k$ (see equation (3)). This will help understand the complex economic interactions at play. Thereby, the analytical model here can be seen as an illustration. However, the nature of the results would be the same with general functions unless the model obtained is more cumbersome and adds intractability (i.e. is only numerically solvable).

3.1 Environment

Let $\mathcal{X} = [0, 1]$ be a linear and closed city composed of a continuum of locations denoted by $x \in \mathcal{X}$. The city is monocentric: $x = 0$ is the Central Business District (hereafter CBD) where all firms are exogenously located. Accordingly, x also represents distance to city center and access to jobs.

3.1.1 Job Matching

The labor market under study gathers a continuum of (*ex ante*) homogenous, infinitely lived and risk neutral unemployed with mass $u \in [0, 1]$ and a continuum of (*ex ante*) identical, infinitely lived and risk neutral employed represented by a mass $e = 1 - u \in [0, 1]$.[6] As a result, u (respectively e) stands for the global unemployment (respectively employment) rate. These workers are spatially dispersed into the city following two endogenous

[6]Unemployed workers are job seekers (i.e. no on-the-job search).

distributions $\mu_U, \mu_W \in \mathcal{M}(\mathcal{X})$:

$$\begin{cases} \mu_U : \mathcal{X} \to \mathbb{R}_+ \\ \int_{\mathcal{X}} \mu_U(x)dx = u \\ \mu_W : \mathcal{X} \to \mathbb{R}_+ \\ \int_{\mathcal{X}} \mu_W(x)dx = e = 1 - u \end{cases} \tag{19}$$

where $\mu_U(x)$ is the density of unemployed workers located in x and $\mu_W(x)$ is the density of employees residing in x. There also exists a continuum of vacant jobs with mass $v \in [0, 1]$ where v is referred to as the vacancy rate. Job seekers find a job and vacancies are filled according to two random processes. These processes are governed by a matching function with constant return to scales denoted by $m(\bar{s}u, v)$ with $\bar{s} \in]0, 1]$ the exogenous average search effort of unemployed workers. Hence, in this city, the job filling rate is $q(\theta) = \frac{m(\bar{s}u,v)}{v}$ with $\theta = \frac{v}{\bar{s}u}$ the labor market tightness in effort units and such that $\frac{\partial q(\theta)}{\partial \theta} < 0$. Likewise, the job finding rate is $f(\theta) = \frac{s}{\bar{s}} \frac{m(\bar{s}u,v)}{u} = s\theta q(\theta)$ so that $\frac{\partial f(\theta)}{\partial \theta} > 0$ and where $s = \bar{s}$ is the search effort of unemployed.[7]

3.1.2 Firms

Firms are placed in $x = 0$, consume no space and can remain in two different situations: either productive or unproductive. If a firm is productive, it is associated with a worker residing in location x and makes the following instantaneous profit:

$$J(x) = y - \omega(x) \tag{20}$$

with $y \in \mathbb{R}_+^*$ the worker's productivity, $\omega(x) \in \mathbb{R}_+^*$ the worker's wage and so that, for every x in $Supp(\mu_W)$, $y > \omega(x)$. Since jobs are destroyed according to an exogenous rate $\delta \in \mathbb{R}_+^*$ referred to as the separation rate, the expected profit of a productive firm (i.e. a filled job) that employs a worker located in x denoted by $\mathcal{J}(x)$ satisfies a Bellman equation:

$$\rho\mathcal{J}(x) = J(x) - \delta\left[\mathcal{J}(x) - \mathcal{V}\right] \tag{21}$$

where ρ is a parameter that captures the preference for the present, $\mathcal{V}$ is the expected profit of an unproductive firm and $\mathcal{J}(x) - \mathcal{V}$ is the local firm's surplus. If a firm is unproductive, it is unfilled by a worker. As a consequence, it posts a unique vacancy at cost $\kappa \in \mathbb{R}_+^*$. As the vacant job is filled at rate $q(\theta)$, the instantaneous profit for an unproductive firm (i.e. a vacancy) is:

$$V = -\kappa \tag{22}$$

and the expected profit of an unproductive firm is:

$$\rho\mathcal{V} = V + q(\theta)\left[\mathcal{J}(x) - \mathcal{V}\right] \tag{23}$$

[7] m also complies with Inada's conditions: $\lim_{\theta\to+\infty} f(\theta) = \lim_{\theta\to 0} q(\theta) = +\infty$ and $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to+\infty} q(\theta) = 0$.

3.1.3 Workers

Workers decide how much to consume a place of residence and can remain in two different states: either employed or unemployed. If a worker is unemployed, it is endowed with hyperbolic preferences à la Mossay and Picard (2011, 2013):

$$\mathcal{Z}(\sigma_U(x), \zeta_U(x)) = \sigma_U(x) - \frac{\phi}{2\zeta_U(x)} \tag{24}$$

with $\sigma_U(x) \in \mathbb{R}_+^*$ (respectively $\zeta_U(x) \in \mathbb{R}_+^*$) the amount of composite good (respectively land) consumed by an unemployed person placed in x and $\phi \in \mathbb{R}_+^*$ the preference for land.[8] It also earns a level of benefits $z \in \mathbb{R}_+^*$ such that $y > z$, goes to the CBD to look for a job incurring linear transport costs $st \in \mathbb{R}_+^*$, faces a rate $f(\theta)$ to have a job, pays $R(x)$ per unit of land to absent landlords and bears high relocation costs.[9] In this case, its budget constraint is:

$$\sigma_U(x) + \zeta_U(x)R(x) + stx = z \tag{25}$$

and the expected utility of an unemployed worker in x denoted by $\mathcal{U}(x)$ is determined by the following Bellman equation:

$$\rho\mathcal{U}(x) = \mathcal{Z}(\sigma_U(x), \zeta_U(x)) + f(\theta)\left[\mathcal{W}(x) - \mathcal{U}(x)\right] \tag{26}$$

with $\mathcal{W}(x)$ the expected utility of an unemployed worker located in x and $\mathcal{W}(x) - \mathcal{U}(x)$ the local worker's surplus. If workers are employed, they share the same utility function than the unemployed one:

$$\mathcal{Z}(\sigma_W(x), \zeta_W(x)) = \sigma_W(x) - \frac{\phi}{2\zeta_W(x)} \tag{27}$$

with $\sigma_W(x) \in \mathbb{R}_+^*$ (respectively $\zeta_W(x) \in \mathbb{R}_+^*$) the amount of composite good (respectively land) consumed by an employed person residing in x. They are also endowed with one unit of labor, a level of productivity y, earns a wage $\omega(x)$, faces high relocation costs, a rate δ of losing his job, commutes to the CBD to work incurring linear transport costs t per

[8] Hyperbolic preferences are a special case of quasi-linear preferences and so the income effect in the land consumption is eliminated (see Zenou (2009a) for other models that tackle the question of city configuration and where the income effect is ruled out in the consumption of land). I use these particular preferences for the sake of simplicity. They have the convenient property that the instantaneous indirect utility linearly depends on residential density of workers. This simplification helps derive the important analytical results in this article. In the case of other quasi-linear utilities and non quasi-linear utilities, the model remains true but is bulky so that it is only numerically solvable.

[9] Unemployment benefits is assumed to be exogenously financed by landlords. This assumption could be relaxed without any difficulty.

Following Zenou (2003, 2006, 2009b) and Kawata and Sato (2012), high relocation costs imply that once the agent is located, it sticks on this place forever. However, note that results are unchanged if this assumption is ruled out.

unit of distance and pays $R(x)$ per unit of land to absent landlords. Thence, their budget constraint is:

$$\sigma_W(x) + R(x)\zeta_W(x) + tx + \varphi\mu_W(x) = \omega(x) \tag{28}$$

and the expected utility of an employed worker in x denoted by $\mathcal{W}(x)$ satisfies the following Bellman equation:

$$\rho\mathcal{W}(x) = \mathcal{Z}(\sigma_W(x), \zeta_W(x)) - \delta\left[\mathcal{W}(x) - \mathcal{U}(x)\right] \tag{29}$$

Another difference between the employed and the unemployed workers is that the employed ones bear $\varphi\mu_W(x)$ (with $\varphi > 0$) an additional cost emphasizing a local negative effect. Even if I do not micro-found this neighborhood externality, the latter is quite intuitive. It can be viewed as competition effect in the consumption of congestible city goods or local amenities (see Fujita (1989, Chap 6.5) and the notion of neighborhood goods). It can also be interpreted as a congestion in road (see Chu (1995) and Grauwin and al. (2011) for an example of congestion in road modeled by local densities of agents). Unemployed workers are free of congestion and employed ones do not value unemployed workers as competitors because unemployed workers can use transports or amenities during outside rush hours. These relations are also empirically observed. For example, it is fair to note that the drop in traffic congestion since the 2007 crisis has been due to the large increase in unemployment. Other alternatives could be considered. Among others, I could introduce more complex interactions between workers in the utility function (see Zhang (2004)): $\varphi_W\mu_W(x) - \varphi_U\mu_U(x)$ reflecting the tendency of employed workers to live with other employed and far from unemployed workers in order to benefit from high-quality neighborhood amenities, low crime areas...

3.1.4 Wage Determination

Once the match is made, the firm observes the workers' location, reasonably does not compensate employed workers for intrinsic congestions (i.e. $\varphi = 0$) and the total local surplus $\mathcal{S}(x) = \mathcal{W}(x) - \mathcal{U}(x) + \mathcal{J}(x) - \mathcal{V}$ is negotiated according to a generalized Nash bargaining game:

$$\omega(x) = argmax\left[\mathcal{W}(x) - \mathcal{U}(x)\right]^{\gamma}\left[\mathcal{J}(x) - \mathcal{V}\right]^{1-\gamma} \tag{30}$$

with $0 < \gamma < 1$ the worker's bargaining power. The equilibrium wage stemming from problem (30) is: $\forall x \in Supp(\mu_W)$,

$$\omega(x) = (1-\gamma)\left[z + (1-s)tx\right] + \gamma(y+\kappa) \tag{31}$$

where $z + (1-s)tx$ is the reservation wage and $y + \kappa$ is the outside option. Equation (31) coincides with the one found by Zenou (2009, equation (13)).[10] Therefore, wages decrease

[10] See Boitier and Lepetit (2014) for another micro-foundation of this result.

with the worker's search effort s but increase with unemployment benefits z, the worker's productivity y, the cost of posting a vacant job κ, transports costs t, distance x and the worker's bargaining power γ.[11] The fact that wages increase with distance establishes that firms compensate workers for their spatial costs, which is a well-established empirical fact.[12]

3.1.5 Market Equilibrium $(\zeta_W^*, \zeta_U^*, \mu_W^*, \mu_U^*, \theta^*, u^*)$: An Informal Definition

A market equilibrium is composed of three partial equilibria: a land market equilibrium, a spatial equilibrium and a labor market equilibrium. On land market is obtained an equilibrium housing demand for employed and unemployed workers denoted by ζ_W^* and ζ_U^*. A spatial equilibrium pins down an allocation of employed and unemployed workers in space denoted by μ_W^* and μ_U^*. On labor market is determined a labor market tightness index denoted by θ^* and an unemployment rate denoted by u^*.

3.2 Land Market Equilibrium (ζ_W^*, ζ_U^*): Definition, Existence and Uniqueness

Using equations (24)-(29) yields:

$$\rho\mathcal{W}(x) = \omega(x) - tx - \varphi\mu_W(x) - R(x)\zeta_W(x) - \frac{\phi}{2\zeta_W(x)} - \delta\left[\mathcal{W}(x) - \mathcal{U}(x)\right] \tag{32}$$

and

$$\rho\mathcal{U}(x) = z - stx - R(x)\zeta_U(x) - \frac{\phi}{2\zeta_U(x)} + f(\theta)\left[\mathcal{W}(x) - \mathcal{U}(x)\right] \tag{33}$$

Given this setup, a land market equilibrium is written as:

Definition 2 *A land market equilibrium consists in finding an equilibrium housing demand for the employed ζ_W^* and for the unemployed ζ_U^* that maximizes equation (32) and equation (33).*

Assuming that, demand equals supply on the land market:[13]

$$\zeta_U(x)\mu_U(x) + \zeta_W(x)\mu_W(x) = 1 \tag{34}$$

I obtain that:

[11] I reasonably assume that: $y + \kappa > z + (1-s)tx$, $\forall x \in Supp(\mu_W^*)$.

[12] See Madden (1985), Zax (1991) and Barber (1998).

[13] I suppose that land intensity equals to 1 and the agricultural rent equals to 0. These assumptions are standard in urban labor economics. This does not determine the nature of the results.

Proposition 3 *For a given spatial equilibrium* (μ_W^*, μ_U^*) *and for a given labor market equilibrium* (θ^*, u^*)*, the equilibrium housing demand is:*

$$\zeta_W^*(x) = \zeta_U^*(x) = \sqrt{\frac{\phi}{2R(x)}} \tag{35}$$

with

$$R(x) = \frac{\phi}{2}\left[\mu_W^*(x) + \mu_U^*(x)\right]^2 \tag{36}$$

The employment status does not affect the equilibrium land consumption. This is why workers are endowed with the same preferences. Furthermore, since preferences are hyperbolic, note that the equilibrium housing demand is independent of the net income of workers and only relies positively on the preference for land ϕ and negatively on the rent per unit of land $R(x)$. This implies that the equilibrium rent per unit of land is only based on the preference for land ϕ and the residential density $\mu_U^*(x) + \mu_W^*(x)$.

3.3 Spatial Equilibium (μ_W^*, μ_U^*)

3.3.1 Definition, Existence and Uniqueness

Plugging solutions of the land market equilibrium (35)-(36) and wage equation (31) in Bellman equations (32) and (33), it gives:

$$\rho\mathcal{W}(x) = (1-\gamma)z + \gamma(y+\kappa) - \left[1-(1-\gamma)(1-s)\right]tx - (\phi+\varphi)\mu_W(x) - \phi\mu_U(x) - \delta\mathcal{E}(x) \tag{37}$$

and

$$\rho\mathcal{U}(x) = z - stx - \phi\mu_W(x) - \phi\mu_U(x) + f(\theta)\mathcal{E}(x) \tag{38}$$

with $\mathcal{E}(x) = \mathcal{W}(x) - \mathcal{U}(x)$. This stresses the presence of a trade-off from the point of view of workers. The latter is synthesized by the decreasing relationships between utilities $\mathcal{W}$ and $\mathcal{U}$ and both x, the distance, and $\mu_W(x)$, $\mu_U(x)$ the residential densities of employed and unemployed workers. This respectively expresses the will to reside close to the CBD and the will to escape congestion. In fact, workers want to live in the most attractive locations in order to avoid paying high transport costs, but at the same time, because they share the same incentive, they anticipate that these places of residence will be coveted by their competitors. To escape competition, workers are encouraged to play differently by residing to more remote locations. Put differently, one agglomeration force, related with transport costs, and two dispersion forces, from congestion in the neighborhood, are at play. Nonetheless, these forces variously impact the employed and the unemployed workers. Indeed, workers do not have the same incentives to cluster around the city center. Unemployed workers minimize their transport costs stx whereas employed workers minimize them $\left[1-(1-\gamma)(1-s)\right]tx$. Analogously, workers do not have the same motives to disperse throughout space. Employed workers disperse because of rent prices $\phi\mu_W(x)+\phi\mu_U(x)$ and of asymmetric congestion $\varphi\mu_W(x)$ while unemployed workers disperse only because of rents $\phi\mu_W(x) + \phi\mu_U(x)$. Thus, the location of workers is driven by their distance from the city center and their neighborhood composition. In other words, workers are strategic since they select a residential location in accordance with their preferences and the strategies of others captured by the endogenous densities $\mu_U(x)$ and $\mu_W(x)$. Therefore, the suited spatial equilibrium is a Nash equilibrium. With infinite number of agents and interactions through densities, it has been shown that the Nash equilibrium takes the following form:

Definition 3 *A spatial equilibrium is a vector* $(\mu_W^*, \mu_U^*) \in \mathcal{M}(\mathcal{X})^2$ *if, and only if:*

$$\begin{cases} Supp(\mu_W^*) \subset arg\max_{x\in\mathcal{X}} \mathcal{W}(x) \\ Supp(\mu_U^*) \subset arg\max_{x\in\mathcal{X}} \mathcal{U}(x) \end{cases} \tag{39}$$

Comparably to Definition 1, an equilibrium is summarized by a non-arbitrage condition stating that, in each group, all agents reach the same utility level because the benefit from living near the city center balances neighborhood composition costs. Such a state accommodates individuals because they are indifferent and therefore unilateral deviations of players are impossible.

Proposition 4 *For a given land market equilibrium* (ζ_W^*, ζ_U^*) *and for a given labor market equilibrium* (θ^*, u^*)*, a unique spatial equilibrium* (μ_W^*, μ_U^*) *exists.*

This finding owes its properties to Propositions 1 and 2. More specifically, existence and uniqueness are due to the fact that the continuous payoff functions $\mathcal{W}$ and $\mathcal{U}$ expose asymmetric congestion interactions: players' utilities are negatively correlated to how tough the competition is. This involves that multiplicity is an outcome of this model if, and only if, $\varphi = 0$. Also note that it is sufficient to assume that agents bid for land to end up with the classical urban model, that is:

$$R^*(x) = max\,\{\Psi(\mathcal{W}^*, x), \Psi(\mathcal{U}^*, x), 0\} \tag{40}$$

where $\Psi(\mathcal{W}^*, x)$ (respectively $\Psi(\mathcal{U}^*, x)$) is the bid-rent function of employed (respectively unemployed) workers. Such equilibrium will spawn few city configurations. Namely, a segregated city will occur in equilibrium where the employed workers reside close to their job locations whereas the unemployed ones live on the outskirts of the city.

3.3.2 Closed Form Solution for the Spatial Distribution of Employed Workers μ_W^*

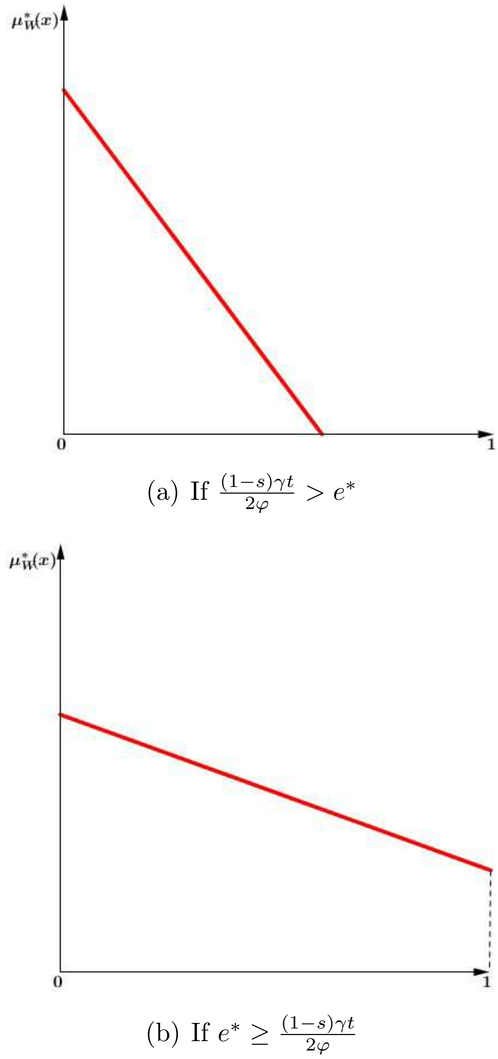
Solving system (39) for μ_W gives:

$$\mu_W^*(x) = \begin{cases} \sqrt{\frac{2(1-s)\gamma te^*}{\varphi}} - \frac{(1-s)\gamma t}{\varphi}x & \text{if } \frac{(1-s)\gamma t}{2\varphi} > e^* \\ e^* + \frac{(1-s)\gamma t}{2\varphi} - \frac{(1-s)\gamma t}{\varphi}x & \text{if } e^* \geq \frac{(1-s)\gamma t}{2\varphi} \end{cases} \tag{41}$$

$\forall x \in Supp(\mu^*_W) = [0, \check{x}^*_W]$ with

$$\check{x}^*_W = \begin{cases} \sqrt{\frac{2\varphi e^*}{(1-s)\gamma t}} & \text{if } \frac{(1-s)\gamma t}{2\varphi} > e^* \\ 1 & \text{if } e^* \geq \frac{(1-s)\gamma t}{2\varphi} \end{cases} \tag{42}$$

Two comments are in order. First, unemployment benefits z, the worker's productivity y, the separation rate δ and the cost of posting a job κ indirectly influence the spatial equilibrium (sketched in Figure 1) via the equilibrium employment rate e^*.[14] This is explained by the fact that, under hyperbolic preferences, the net income effect is cancelled out. Second,



(a) If $\frac{(1-s)\gamma t}{2\varphi} > e^*$

(b) If $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$

Figure 1: Spatial Distribution of Employed Workers μ^*_W

for e^* fixed, the impact of each parameter on the spatial concentration of employed workers is well-established (see Figure 2). An increase in transport costs t incite workers to live

[14] In Figures 1-9, red lines, blue lines and green lines represent respectively employed, unemployed workers and the local unemployment rate.

closer to the city center. Consequently, the CBD is more inhabited and the density gradient is steeper. In addition, a higher φ makes competition tougher between employed workers. This encourages employed workers to escape congestion by living (in proportion) farer to their job locations. The presence of the worker's bargaining power γ and the worker's search effort s are also intuitive. If the worker's bargaining power is improved, employed workers are less compensated by firms for their transport costs (see equation (31)). This reinforces the negative effect in transport costs and leads to a higher urban density of employed workers. If s is larger, unemployed workers go more frequently to the city center and therefore bear more commuting costs. In consequence, they agglomerate near the city center.

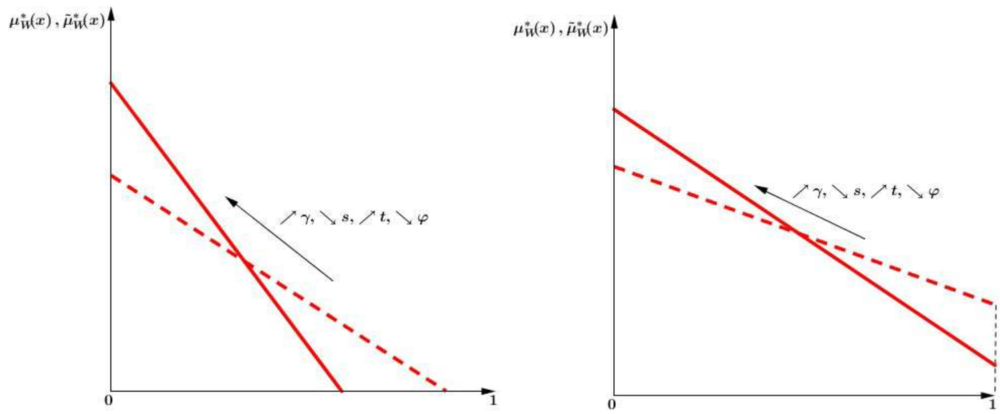


Figure 2: Effects of the Parameters on the Spatial Distribution of Employed Workers

3.3.3 Closed Form Solution for the Spatial Distribution of Unemployed Workers μ_U^*

By the same token, solving system (39) for μ_U, I find that, if:

$$\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0 \tag{43}$$

then

$$\mu_U^*(x) = \begin{cases} \sqrt{2\left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] t(1-e^*)} - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] tx & \text{if } e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] \frac{t}{2} \\ 1 - e^* + \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] \frac{t}{2} - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] tx & \text{if } 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] \frac{t}{2} \geq e^* \end{cases} \tag{44}$$

$\forall x \in [0, \check{x}_U^*]$ with

$$\check{x}_U^* = \begin{cases} \sqrt{\frac{2\phi\varphi(1-e^*)}{[\varphi s - (1-s)\gamma\phi]t}} & \text{if } e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] \frac{t}{2} \\ 1 & \text{if } 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right] \frac{t}{2} \geq e^* \end{cases} \tag{45}$$

In contrast, if:

$$\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0 \tag{46}$$

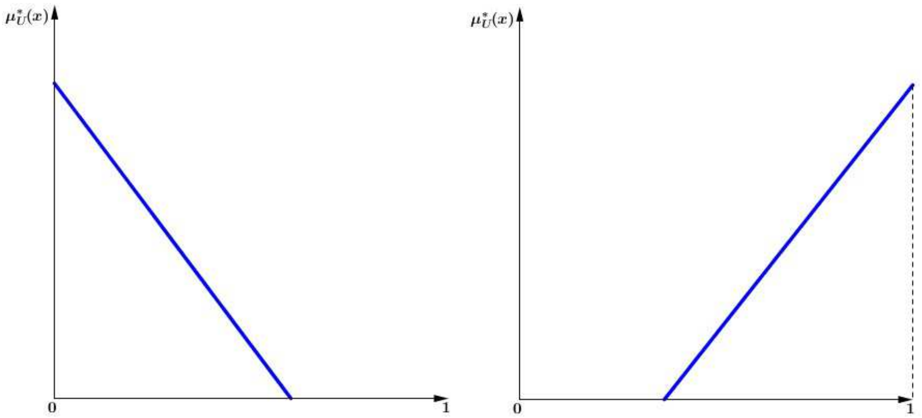
then

$$\mu_U^*(x) = \begin{cases} 1 - \sqrt{2\left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right] t(1-e^*) + \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right] tx} & \text{if } e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \\ 1 - e^* + \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} + \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right] x & \text{if } 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^* \end{cases} \tag{47}$$

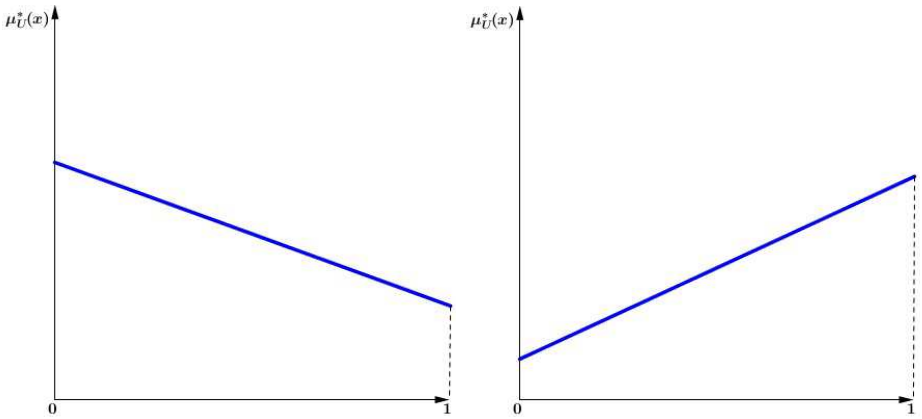
$\forall x \in [\check{x}_U^*, 1]$ with

$$\check{x}_U^* = \begin{cases} \sqrt{\frac{2\phi\varphi(1-e^*)}{[(1-s)\gamma\phi - \varphi s]t}} & \text{if } e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \\ 0 & \text{if } 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^* \end{cases} \tag{48}$$

Figure 3 and Figure 4 descibe the behavior of equations (43)-(48). The first comment



(a) If $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ and $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ (b) If $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ and $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$



(c) If $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ and $1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2} \geq e^*$ (d) If $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ and $1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^*$

Figure 3: Spatial Distribution of Unemployed Workers μ_U^*

made in the previous section also applies here. Another interesting result lies in the connection of μ_U^* and distance to jobs x: the unemployed workers' distribution can decrease or counter-intuitively increase with respect to distance. The predominance of one of these two situations is determined by condition (43) that captures the relative agglomeration and dispersion forces of workers. It particularly states the interdependence between the location of employed workers and the location of unemployed ones: $\frac{s}{\phi}$ can be viewed as the relative incentive of unemployed workers, while $\frac{(1-s)\gamma}{\varphi}$ can be interpreted as the relative motive of employed ones. In fact, the worker's bargaining power γ, the worker's search effort s and the specific congestion effect φ play similar roles found in Section 3.3.2. If γ is high, s and φ are low enough, the employed concentrate near the CBD and the unemployed flee the center to live on the outskirts. Furthermore, when the preference for land ϕ is high, workers want to increase their housing consumptions. Because the land supply is fixed, they know that if they reside in dense areas, the land prices will be high and they will not be able to achieve their willingness to live in large dwellings. This phenomenon exhorts workers to disperse: the unemployed desert areas where employed workers stay. Last, transport costs t only magnify the previously mentioned forces.

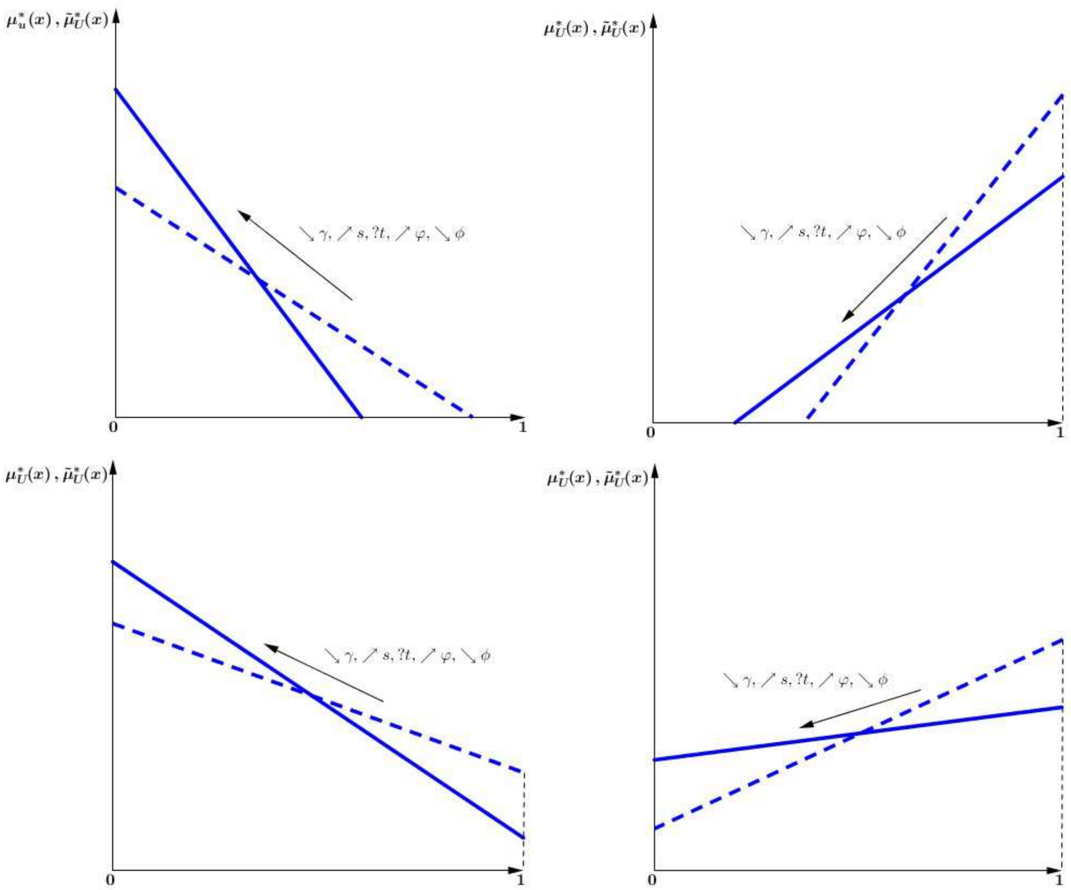


Figure 4: Effects of the Parameters on the Spatial Distribution of Unemployed Workers

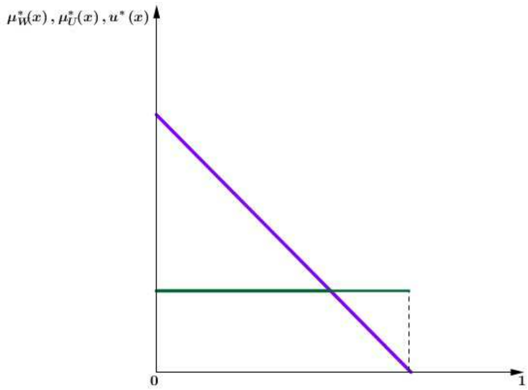
3.4 Unemployment Dispersion and City Configurations

Let $u(x)$ be the local unemployment rate defined as:

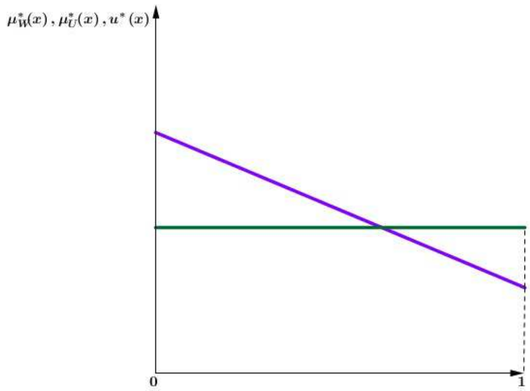
$$u(x) = \frac{\mu_U^*(x)}{\mu_U^*(x) + \mu_W^*(x)} \tag{49}$$

Combining results outlined in Section 3.3 and definition (49), it turns out that the model generates many different levels of unemployment dispersion. For clarity of exposition, I rank this large heterogeneity into 5 patterns and 6 sub-patterns:

(i)- a city is said to be purely integrated if both groups of workers live together and in the same proportions. Thereby, the local unemployment rate is uniformly distributed throughout space (see Figure 5).



(a) If **C1a** holds



(b) If **C1b** holds

Figure 5: Examples of Purely Integrated Cities

(ii)- a city is segregated if populations of workers get separated into two different parts of the city, that is, if the distribution of the local unemployment rate is degenerated. Thus, there are only two local unemployment rates in the economy: either 0% or 100%. Because of the fact that spatial distribution of employed workers always decreases with respect to jobs, the developed framework only supports a situation where the employed live close to jobs and the unemployed reside at the fringes of the city (see Figure 6).

(iii)- a city is said to be integrated when employed and unemployed workers share the same place of residence but in different proportions. Based on this feature, the local unemployment rate is continuously (but non-uniformly) dispersed over space (see Figure 7).

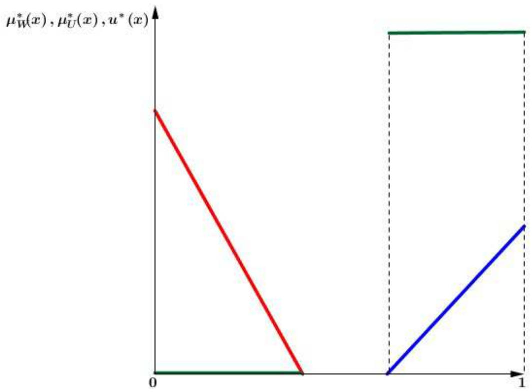
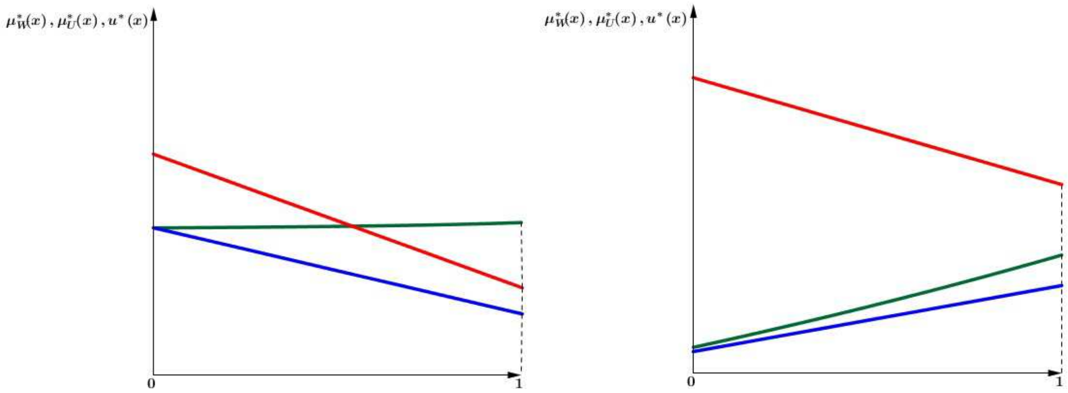


Figure 6: Example of Segregated Cities



(a) If **C3a** holds

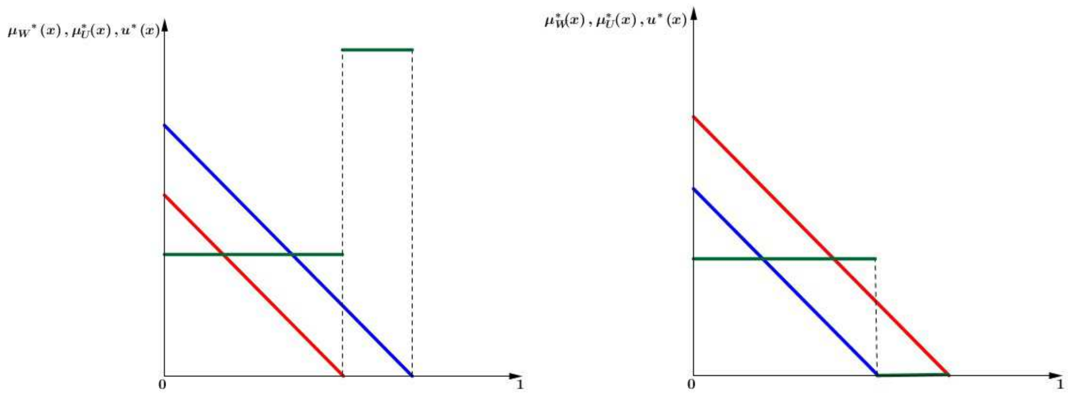
(b) If **C3b** holds

Figure 7: Examples of Integrated Cities

(iv)- a city is incompletely and purely integrated if the local unemployment rate is uniformly diffused into a part of the space, whereas in other zones of the city, it is degenerated (see Figure 8). Two sub-patterns can be pointed out: a situation where the city exposes a segregated are of unemployed (see Figure 8a) or of employed workers (See Figure 8b).

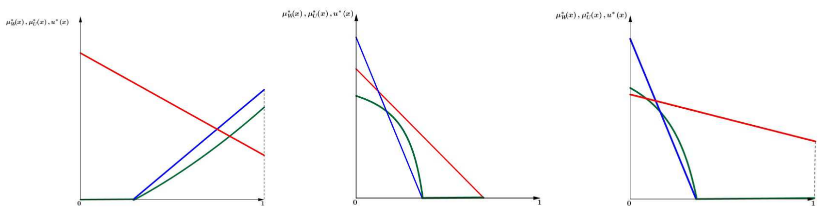
(v)- a city is said to be incompletely integrated (or incompletely segregated) if the local unemployment rate is continuously diffused into a zone of the city while in other areas, it is degenerated. Four sub-patterns can be underlined: a central core of employed workers surrounded by a peripheral integrated ring of workers (see Figure 9a), a central core of mixed workers surrounded by a peripheral segregated part of the employed (see Figures 9b-9c) or the unemployed (see Figure 9d-9f) and both a central core and a peripheral ring of segregated areas separated by an intermediate ring of mixed workers (see Figure 9g).

The prevalence of one these city configurations hinges on the relative size of agglomeration and dispersion forces (i.e. comparing the slopes of the spatial distributions of workers).

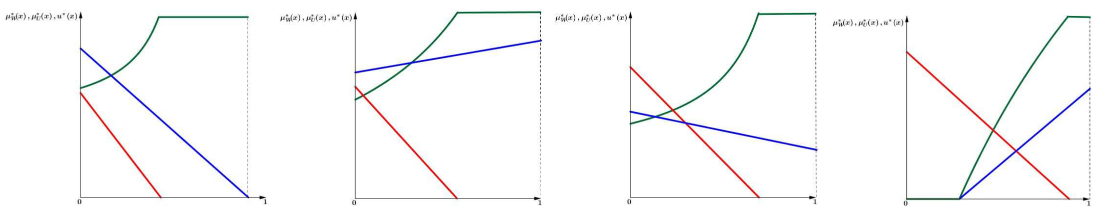


(a) If **C4a** holds (b) If **C4b** holds

Figure 8: Examples of Incompletely and Purely Integrated Cities



(a) If **C5a** holds (b) If **C5b** holds (c) If **C5c** holds



(d) If **C5d** holds (e) If **C5e** holds (f) If **C5f** holds (g) If **C5g** holds

Figure 9: Examples of Incompletely Integrated Cities

Indeed, a deeper analysis of equations (42)-(49) yields:

Proposition 5 *For a given market equilibrium* $(\zeta_W^*, \zeta_U^*, \mu_W^*, \mu_U^*, \theta^*, u^*)$, *if:*

C1a: $e^* = \frac{1}{2}$, $\frac{(1-s)\gamma t}{2\varphi} > \frac{1}{2} > 1 - \frac{(1-s)\gamma t}{2\varphi}$ *and* $\frac{s}{\phi} = \frac{2(1-s)\gamma}{\varphi}$ *then the city is purely integrated (see Figure 5a).*

C1b: $e^* = \frac{1}{2}$, $\frac{1}{2} \geq \frac{(1-s)\gamma t}{2\varphi}$, $1 - \frac{(1-s)\gamma t}{2\varphi} \geq \frac{1}{2}$ *and* $\frac{s}{\phi} = \frac{2(1-s)\gamma}{\varphi}$ *then the city is purely integrated (see Figure 5b).*

C2: $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *and* $\frac{(1-s)\gamma\phi}{2(1-s)\phi - \varphi s} \geq e^*$ *then the city is segregated (see Figure 6).*

C3a: $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ *and* $1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2} \geq e^*$ *then the city is integrated (see Figure 7.a).*

C3b: $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ *and* $1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^*$ *then the city is integrated (see Figure 7.b).*

C4a: $e^* < \frac{1}{2}$ *and* $\frac{s}{\phi} = \frac{2(1-s)t}{\varphi}$ *then the city is incompletely and purely integrated (see Figure 8a).*

C4b: $e^* > \frac{1}{2}$ *and* $\frac{s}{\phi} = \frac{2(1-s)t}{\varphi}$ *then the city is incompletely and purely integrated (see Figure 8b).*

C5a: $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$ *and* $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *then the city is incompletely integrated (see Figure 9a).*

C5b: $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $e^* > \frac{(1-s)\gamma\phi}{\varphi s}$ *then the city is incompletely integrated (see Figure 9b).*

C5c: $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$ *and* $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *then the city is incompletely integrated (see Figure 9c).*

C5d: $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $e^* < \frac{(1-s)\gamma\phi}{\varphi s}$ *then the city is incompletely integrated (see Figure 9d).*

C5e: $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ *and* $1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^*$ *then the city is incompletely integrated (see Figure 9e).*

C5f: $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ *and* $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *then the city is incompletely integrated (see Figure 9f).*

C5g: $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *and* $e^* > \frac{(1-s)\gamma\phi}{2(1-s)\gamma\phi - s\varphi}$ *then the city is incompletely integrated (See Figure 9g).*

What is appealing is that very simple analytical conditions for the 11 different urban situations are obtained. This shows that the present model is able to explain why cities are different in terms of unemployment disparities with parsimonious parameters. Namely, purely integrated and incompletely integrated cities emerge when the relative incentive of employed workers $\frac{(1-s)\gamma}{\varphi}$ is two times higher than the one of unemployed workers $\frac{s}{\phi}$.

Segregated cities or incompletely integrated cities that converge towards segregated cities (i.e. Figure 9d) exist if the bargaining power γ, transport costs t and preferences for land ϕ and if search effort s and competition between employed workers φ are low. On the contrary, integrated cities prevail if agglomeration forces are weak (i.e. γ, t low and s large) and dispersion forces are strong (ϕ and φ large). Incompletely integrated cities are a go between segregated and integrated cities.

3.5 Labor Market Equilibrium (θ^*, u^*)

Let me close now the model by computing e^* the equilibrium employment rate of the city.

3.5.1 Modified Beveridge Curve

The dynamics of the global unemployment rate u is:

$$\dot{u} = \int_{Supp(\mu_W^*)} \delta \mu_W^*(x)dx - \int_{Supp(\mu_U^*)} f(\theta)\mu_U^*(x)dx = \delta(1-u) - f(\theta)u \tag{50}$$

with $\dot{u}$ the variation of unemployment with respect to time, $\delta(1-u)$ is the number of employed workers entering unemployment and $f(\theta)u$ is the number of unemployed workers finding a job. In steady state, the flows are equal so that:

$$u = \frac{\delta}{\delta + f(\theta)} = \frac{\delta}{\delta + s\theta q(\theta)} \tag{51}$$

Equation (51) is referred as the modified Beveridge curve. As in Pissarides (2000), this curve shows an inverse relationship between the unemployment rate u and the vacancy rate u and the unemployment rate u increases with the separation rate δ. Similarly to Zenou (2009b), an increase in search efforts s lowers unemployment.

3.5.2 Average Wage Equation

Unproductive firms do not know the location of their future workers when they post their vacancies but make their decisions by expecting the average wage denoted by $\overline{\omega}$ and defined as:

$$\overline{\omega} = \frac{1}{e^*} \int_{Supp(\mu_W^*)} \omega^*(x)\mu_W^*(x)dx \tag{52}$$

or, using equilibrium wage equation (31) and equilibrium spatial distribution (41):

$$\overline{\omega} = \begin{cases} (1-\gamma)\left[z + \frac{1}{3}\sqrt{\frac{2(1-s)\varphi t e^*}{\gamma}}\right] + \gamma(y+\kappa) & \text{if } \frac{(1-s)t}{2\varphi} > e^* \\ (1-\gamma)\left[z + (1-s)t\left(\frac{1}{2} - \frac{(1-s)\gamma t}{12\phi e^*}\right)\right] + \gamma(y+\kappa) & \text{if } e^* \geq \frac{(1-s)t}{2\varphi} \end{cases} \tag{53}$$

Unemployed benefits z, the worker's productivity y and the cost of a vacancy κ share the same role on the average wage $\overline{\omega}$ than on the wage equation $\omega(x)$. The worker's bargaining

γ has a positive impact on the average wage if, and only if, $y + \kappa > z + (1-s)t\frac{\partial \overline{x}}{\partial \gamma}$. The average wage is improved by transport costs t, the congestion effect φ and the worker's search effort s since the latter increases average distance between jobs and workers and lowers the compensation mechanism previously described.

3.5.3 Modified Job Creation Equation

Let $\overline{\mathcal{J}}$ be the expected asset value of a filled job. This asset value follows a Bellman equation:

$$\rho\overline{\mathcal{J}} = y - \overline{\omega} - \delta\left(\overline{\mathcal{J}} - \mathcal{V}\right) \tag{54}$$

Using equation (53), equation (54) and the free entry condition (i.e. $\mathcal{V} = 0$)), the modified job creation equation is:

$$\frac{y - \overline{\omega}}{\rho + \delta} = \frac{\kappa}{q(\theta)} \tag{55}$$

This standard equation, stating an inverse relation between the labor market tightness and the average wage, has the straightforward following explanation. In equilibrium, the average benefit of a filled job (i.e. the benefit of a filled job multiplied by the expected average duration of a filled job) is equal to the average search cost of a vacancy (i.e. the cost of a vacancy multiplied by the average duration of a job vacant). Also observe that the local unemployment rate $u^*(x)$ and preferences for land ϕ do not influence the level of labor market tightness and so the unemployment rate u. This means that the intra-variation of the local unemployment rate does not impact the global unemployment rate of the city. Only the spatial concentration of employed workers $\mu_W^*(x)$ matters with the average wage. More precisely, urban density of the employed negatively affects the unemployment rate. Lower density increases the average reservation wage since workers live further away from jobs. For that reason, the average wage set by firms is improved and the expected gain of hiring a new worker decreases. To restore the equilibrium, the expected cost of looking for a job has to decrease too. This leads to a fall in labor market tightness and so to an increase in the unemployment rate.[15] A direct corollary of this finding is that segregated cities always show lower unemployment rates than integrated ones.

3.5.4 Definition, Existence and Uniqueness

Definition 4 *A labor market equilibrium (θ^*, u^*) consists in finding a labor market tightness index θ^* solving the modified job creation equation (55) and an unemployment rate u^* solving the modified Beveridge curve (51).*

[15]The reader should keep this intuitive mechanism in mind to understand the relationships in Section 3.5.5.

Proposition 6 *For a given land market equilibrium* (ζ_W^*, ζ_U^*) *and a given spatial equilibrium* (μ_W^*, μ_U^*)*, a unique labor market equilibrium* (θ^*, u^*) *exists.*

3.5.5 Comparive Statics Analysis

In order to get a better rationale of the model, let me perform a comparative statics analysis. Results of this mere exercise are displayed in Figures 10-13. Since unemployment benefits, specific congestion effect and transport costs improve the average reservation wage of workers, this implies a decline in labor market tightness and a higher unemployment rate. Equivalently, an increase in the costs of posting a vacancy leads to greater expected average costs. By definition, this reduces job creation and increases the unemployment rate.

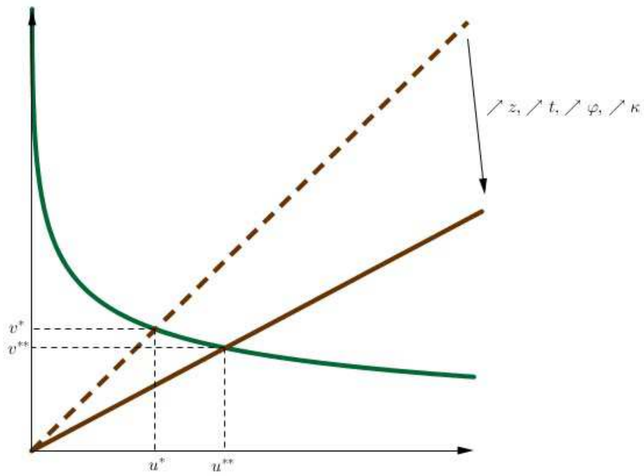


Figure 10: Effects of unemployment benefits, specific congestion effect, transport costs and the cost of posting a vacancy on equilibrium labor market tightness and equilibrium unemployment

The worker's productivity seems to play an ambiguous role: it raises wage pressure, but it also improves production. Nonetheless, the net effect on the expected average profit is positive since $\gamma < 1$ and an increase in the productivity lowers the unemployment rate.

On the one hand, worker's search effort improves the finding rate suggesting that workers will experience a shorter unemployment spell . On the other hand, this parameter lowers the average wage as stated earlier. Thus, the worker's search effort has a positive effect on the labor market tightness, as well as on the unemployment rate.

The effects of an increase in the separation rate and the preference for the present are the same. This diminishes the expected gain of hiring a worker, the labor tightness index is lower and the equilibrium unemployment is higher. To loop the loop, the impact of the bargaining power of workers is uncertain, which is a fairly robust result in urban labor economics (see Zenou (2009b)).

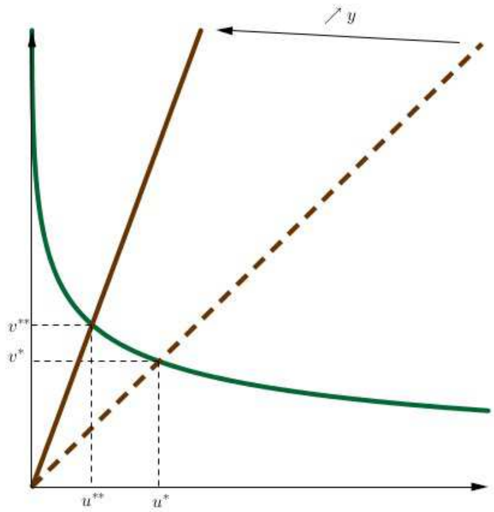


Figure 11: Effects of worker's productivity on equilibrium labor market tightness and equilibrium unemployment

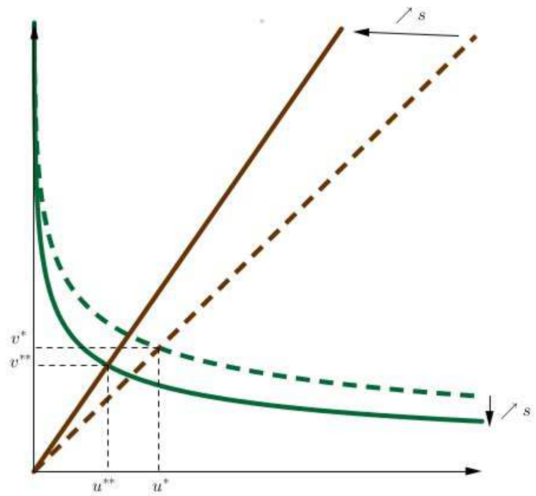


Figure 12: Effects of the worker's search effort on equilibrium labor market tightness and equilibrium unemployment

3.6 Market Equilibrium: $(\zeta_W^*, \zeta_U^*, \mu_W^*, \mu_U^*, \theta^*, u^*)$: Definition, Existence and Uniqueness

Definition 5 *A market equilibrium $(\zeta_W^*, \zeta_U^*, \mu_W^*, \mu_U^*, \theta^*, u^*)$ is such that a land market equilibrium ζ^*, a spatial equilibrium μ^* and a labor market equilibrium (θ^*, u^*) are solved for simultaneously.*

Proposition 7 *A unique market equilibrium $(\zeta_W^*, \zeta_U^*, \mu_W^*, \mu_U^*, \theta^*, u^*)$ exists if, and only if:*

$$(1-\gamma)(y-z)+\gamma\kappa > \frac{(1-s)}{3}\sqrt{\frac{2(1-s)\varphi t e^*}{\gamma}} > (1-\gamma)(1-s)t\left(\frac{1}{2}-\frac{(1-s)\gamma t}{12\varphi e^*}\right) \quad (56)$$

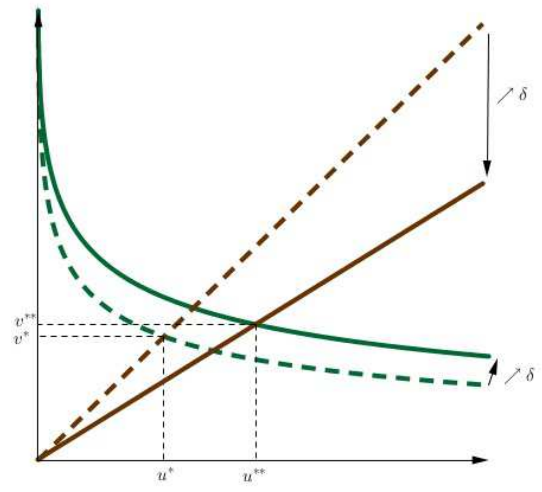


Figure 13: Effects of the separation rate on equilibrium labor market tightness and equilibrium unemployment

4 Conclusion

Although cities show large heterogeneity in terms of spatial variation in the unemployment rate, the theoretical literature focuses on the analysis of segregated cities where two unemployment rates exist: either 0 % or 100 %. This is because land is determined by the bid-rent theory that precludes cases where unemployed and employed workers live together. The purpose of this paper is twofold. First, I highlight the main theoretical features of a general urban model where agents do not bid for land. In this simple model, the allocation of workers throughout space is driven by a Nash equilibrium. I prove that such equilibrium exists and is unique if, and only if, workers' utility functions display asymmetric local congestion interactions. Second, I use the previous setting to account for unemployment dispersion heterogeneity within cities. In equilibrium, many new city configurations can emerge where the local unemployment rate exhibits different movements. Therefore, the obtained model easily matches many city patterns observed in the data. I then determine the conditions under which each configuration can prevail. I also prove the existence and the uniqueness of a labor market equilibrium and make the connection of the latter with the spatial distributions of workers.

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5 Proof

Proofs 1-2 are inspired from Cardaliaguet (2012) and Cirant (2014).

Proof 1 *Assume that $Y = \mathcal{M}(\mathcal{X})$ and let $\Gamma : Y \times Y \to 2^Y \times 2^Y$ be defined by:*

$$\Gamma_1(\mu_1, \mu_2) = arg \max_{\mu_1^* \in Y} \int_{\mathcal{X}} \mathcal{A}_1(x, \mu_1(x), \mu_2(x)) d\mu_1^*(x),$$
$$\Gamma_2(\mu_1, \mu_2) = arg \max_{\mu_2^* \in Y} \int_{\mathcal{X}} \mathcal{A}_2(x, \mu_1(x), \mu_2(x)) d\mu_2^*(x),$$

Γ is upper-semicontinuous multi-application with convex compact values. Furthermore, $Y \times Y$ is a convex compact set of a locally convex Hausdroff space. This implies that, by Ky Fan fixed point theorem, Γ admits a fixed point: $\exists (\mu_1^*, \mu_2^*) \in Y \times Y$ such that: for $k = 1, 2$,

$$\int_{\mathcal{X}} \mathcal{A}_k \left(x, \mu_1^*(x), \mu_2^*(x)\right) d\mu_k^*(x) = \sup_{\mu \in Y} \int_{\mathcal{X}} \mathcal{A}_k \left(x, \mu_1^*(x), \mu_2^*(x)\right) d\mu(x)$$

Proof 2 *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the following condition:*

$$\int_{\mathcal{X}} \mathcal{A}_1 \left(x, \mu_1^*(x), \mu_2^*(x)\right) - \mathcal{A}_1 \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d \left(\mu_1^* - \tilde{\mu}_1\right)(x) +$$
$$\int_{\mathcal{X}} \mathcal{A}_2 \left(x, \mu_1^*(x), \mu_2^*(x)\right) - \mathcal{A}_2 \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d \left(\mu_2^* - \tilde{\mu}_2\right)(x) < 0$$

for all $\mu_1^* \neq \tilde{\mu}_1$ or $\mu_2^* \neq \tilde{\mu}_2$. Using Definition 1, it comes that:

$$\int_{\mathcal{X}} \mathcal{A}_1 \left(x, \mu_1^*(x), \mu_2^*(x)\right) d\mu_1^*(x) \geq \int_{\mathcal{X}} \mathcal{A}_1 \left(x, \mu_1^*(x), \mu_2^*(x)\right) d\tilde{\mu}_1(x)$$

and

$$\int_{\mathcal{X}} \mathcal{A}_1 \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d\tilde{\mu}_1(x) \geq \int_{\mathcal{X}} \mathcal{A}_1 \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d\mu_1^*(x)$$

By subtracting both equations above, I obtain:

$$\int_{\mathcal{X}} \mathcal{A}_1 \left(x, \mu_1^*(x), \mu_2^*(x)\right) - \mathcal{A}_1 \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d \left(\mu_1^* - \tilde{\mu}_1\right)(x) \geq 0$$

By the same token, I get:

$$\int_{\mathcal{X}} \mathcal{A}_2 \left(x, \mu_1^*(x), \mu_2^*(x)\right) - \mathcal{A}_2 \left(x, \tilde{\mu}_1(x), \tilde{\mu}_2(x)\right) d \left(\mu_2^* - \tilde{\mu}_2\right)(x) \geq 0$$

Using the first assumption, this implies that $\mu_1^* = \tilde{\mu}_1$ and $\mu_2^* = \tilde{\mu}_2$.

Proof 3 *From equations (24)-(33), I have:*

$$\mathcal{W}(x) = \frac{[r + f(\theta)] \left[\omega(x) - tx - \varphi\mu_W(x)\right] + \delta(z - stx)}{r + \delta + f(\theta)} - R(x)\zeta(x) - \frac{\phi}{2\zeta(x)}$$

and

$$\mathcal{U}(x) = \frac{(r+\delta)(z - stx) + f(\theta)\left[\omega(x) - tx - \varphi\mu_W(x)\right]}{r+\delta+f(\theta)} - R(x)\zeta(x) - \frac{\phi}{2\zeta(x)}$$

The solution of Definition 2 is:

$$\zeta_W^*(x) = \zeta_U^*(x) = \sqrt{\frac{\phi}{2R(x)}}$$

that is

$$\zeta_U^*(x)\mu_U^*(x) + \zeta_W^*(x)\mu_W^*(x) = \left[\mu_W^*(x) + \mu_U^*(x)\right]\sqrt{\frac{\phi}{2R(x)}} = 1$$

and so

$$R(x) = \frac{\phi}{2}\left[\mu_W^*(x) + \mu_U^*(x)\right]^2 \geq 0$$

Proof 4 *Trivial using Section 2.*

Proof 5 *By definition:*

- *a purely integrated city exists if:*

- $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$, $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $\sqrt{\frac{2(1-s)\gamma t e^*}{\varphi}} - \frac{(1-s)\gamma t}{\varphi} = \sqrt{2\left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]t(1-e^*)} - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]tx$ *that is if* $e^* = \frac{1}{2}$, $\frac{s}{\phi} = \frac{2(1-s)t}{\varphi}$ *and* $\frac{(1-s)\gamma t}{2\varphi} > \frac{1}{2} > 1 - \frac{(1-s)\gamma t}{2\varphi}$ *(see **C1a**).*

- $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$, $1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2} \geq e^*$ *and* $\sqrt{\frac{2(1-s)\gamma t e^*}{\varphi}} - \frac{(1-s)\gamma t}{\varphi} = \sqrt{2\left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]t(1-e^*)} - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]tx$ *that is if* $e^* = \frac{1}{2}$, $\frac{s}{\phi} = \frac{2(1-s)t}{\varphi}$, $\frac{1}{2} \geq \frac{(1-s)\gamma t}{2\varphi}$ *and* $1 - \frac{(1-s)\gamma t}{2\varphi} \geq \frac{1}{2}$ *(see **C1b**).*

- *a segregated cities emerges (see **C2**) if:* $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$, $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *and* $\sqrt{\frac{2\varphi e^*}{(1-s)\gamma t}} \leq \sqrt{\frac{2\varphi\phi(1-e^*)}{[(1-s)\gamma\phi - \varphi s]t}}$, *or equivalently,* $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *and* $\frac{(1-s)\gamma\phi}{2(1-s)\gamma\phi - s\varphi} \geq e^*$.

- *an integrated city emerges if:*

- $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ *and* $1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2} \geq e^*$ *(see **C3a**).*

or

- $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ *and* $1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^*$ *(see* ***C3b****).*

- *an incompletely purely integrated city exists if:*

- $e^* > \frac{1}{2}$ *and* $\frac{s}{\phi} = \frac{2(1-s)t}{\varphi}$ *(see* ***C4a****).*

or

- $e^* < \frac{1}{2}$ *and* $\frac{s}{\phi} = \frac{2(1-s)t}{\varphi}$ *(see* ***C4b****).*

- *an incompletely integrated city where there is a central core of employed surrounded by a peripheral integrated ring of workers prevails (see* ***C5a****) if:* $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ *and* $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$, *that is, if* $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$ *and* $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$.

- *an incompletely integrated city where there is a central core of mixed ring workers surrounded by a peripheral segregated part of the employed occurs if:*

- $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$, $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $\sqrt{\frac{2\varphi e^*}{(1-s)\gamma t}} > \sqrt{\frac{2\varphi\phi(1-e^*)}{[\varphi s-(1-s)\gamma\phi]t}}$, *that is, if* $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $e^* > \frac{(1-s)\gamma\phi}{\varphi s}$ *(see* ***C5b****).*

or

- $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ *and* $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$, *this to say, if* $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$ *and* $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *(see* ***C5c****).*

- *an incompletely integrated city where there is a central core of mixed ring workers surroundeed by a peripheral segregated part of the unemployed appears if:*

- *if* $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$, $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $\sqrt{\frac{2\varphi e^*}{(1-s)\gamma t}} < \sqrt{\frac{2\varphi\phi(1-e^*)}{[\varphi s-(1-s)\gamma\phi]t}}$, *that is, if* $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *and* $e^* < \frac{(1-s)\gamma\phi}{\varphi s}$ *(see* ***C5d****).*

or

- *if* $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$ *and* $1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2} \geq e^*$ *(see* ***C5e****).*

or

- *if* $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi} > 0$ *and* $e^* > 1 - \left[\frac{s}{\phi} - \frac{(1-s)\gamma}{\varphi}\right]\frac{t}{2}$ *(see* ***C5f****).*

- *an incompletely integrated city where there are both a central core and a peripheral ring of segregated areas separated by an intermediate ring of mixed workers emerges (see **C5g**) if:* $\frac{(1-s)\gamma t}{2\varphi} > e^*$, $\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi} > 0$, $e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *and* $\sqrt{\frac{2\varphi e^*}{(1-s)\gamma t}} > \sqrt{\frac{2\varphi\phi(1-e^*)}{[(1-s)\gamma\phi - \varphi s]t}}$, *or equivalently,* $\frac{(1-s)\gamma t}{2\varphi} > e^* > 1 - \left[\frac{(1-s)\gamma}{\varphi} - \frac{s}{\phi}\right]\frac{t}{2}$ *and* $e^* > \frac{(1-s)\gamma\phi}{2(1-s)\gamma\phi - s\varphi}$. $\diamond$

Proof 6 *I find:*

- *If* $\frac{(1-s)\gamma t}{2\varphi} > e^*$ *then:*

$$\frac{(1-\gamma)(y-z) - \gamma\kappa - \frac{(1-\gamma)}{3}\sqrt{\frac{2(1-s)\varphi t}{\gamma}\frac{f(\theta)}{\delta+f(\theta)}}}{\rho+\delta} = \frac{\kappa}{q(\theta)}$$

Let g_1 *be a continuous function on* $\mathbb{R}_+$ *defined as:*

$$g_1(\theta) = \frac{\kappa}{q(\theta)}$$

Notice that

$$\begin{cases} g_1(0) = 0 & \textit{because } \lim_{\theta\to 0} q(\theta) = +\infty \\ \lim_{\theta\to+\infty} g_1(\theta) = +\infty & \textit{because } \lim_{\theta\to+\infty} q(\theta) = 0 \\ \frac{\partial g_1(\theta)}{\partial\theta} > 0 & \textit{because } \frac{\partial q(\theta)}{\partial\theta} < 0 \end{cases}$$

Let g_2 *be a continuous function on* $\mathbb{R}_+$ *so that:*

$$g_2(\theta) = \frac{(1-\gamma)(y-z) - \gamma\kappa - \frac{(1-\gamma)}{3}\sqrt{\frac{2(1-s)\varphi t}{\gamma}\frac{f(\theta)}{\delta+f(\theta)}}}{\rho+\delta}$$

Observe that

$$\begin{cases} g_2(0) = \frac{(1-\gamma)(y-z)-\gamma\kappa}{\rho+\delta} > 0 & \textit{because } \lim_{\theta\to 0} f(\theta) = 0 \\ \lim_{\theta\to+\infty} g_2(\theta) = \frac{(1-\gamma)(y-z)-\gamma\kappa-\frac{(1-\gamma)}{3}\sqrt{\frac{2(1-s)\varphi t}{\gamma}}}{\rho+\delta} > g_2(0) & \textit{because } \lim_{\theta\to+\infty} f(\theta) = +\infty \\ \frac{\partial g_2(\theta)}{\partial\theta} < 0 & \textit{because } \frac{\partial f(\theta)}{\partial\theta} > 0 \end{cases}$$

The characteristics of functions g_1 *and* g_2 *ensure that a unique and stable labor market tightness index* θ^* *exists. Moreover, if a unique labor market tightness index exists, this implies the existence of a unique unemployment rate denoted by* u^**.*

- *If* $e^* \geq \frac{(1-s)\gamma t}{2\varphi}$ *then:*

$$\frac{(1-\gamma)(y-z) - \gamma\kappa - (1-\gamma)(1-s)t\left\{\frac{1}{2} - \frac{(1-s)\gamma t}{12\phi\frac{f(\theta)}{[\delta+f(\theta)]}}\right\}}{\rho+\delta} = \frac{\kappa}{q(\theta)}$$

Let g_3 be a continuous function on $\mathbb{R}_+$ defined as

$$g_3(\theta) = \frac{(1-\gamma)(y-z) - \gamma\kappa - (1-\gamma)(1-s)t\left\{\frac{1}{2} - \frac{(1-s)\gamma t}{12\phi\frac{f(\theta)}{[\delta+f(\theta)]}}\right\}}{\rho+\delta}$$

Notice that

$$\begin{cases} \lim_{\theta\to 0} g_3(\theta) = +\infty & \textit{because } \lim_{\theta\to 0} f(\theta) = 0 \\ \lim_{\theta\to+\infty} g_3(\theta) = \frac{(1-\gamma)(y-z)-\gamma\kappa-(1-\gamma)(1-s)t\left[\frac{1}{2}-\frac{(1-s)\gamma t}{12\phi}\right]}{\rho+\delta} & \textit{because } \lim_{\theta\to+\infty} f(\theta) = +\infty \\ \frac{\partial g_3(\theta)}{\partial\theta} < 0 & \textit{because } \frac{\partial f(\theta)}{\partial\theta} > 0 \end{cases}$$

The characteristics of functions g_1 and g_3 ensure that a unique labor market tightness index θ^* exists. Furthermore, if a unique labor market tightness index exists, this implies the existence of a unique unemployment rate denoted by u^*. $\diamond$

Proof 7 *As $y > \omega(x)$, $\forall x \in Supp(\mu_W^*)$, a unique general equilibrium $(\zeta_W^*, \zeta_U^*, \mu_W^*, \mu_U^*, \theta^*, u^*)$ exists if, and only if, $y - \omega(\check{x}^*) > 0$ or $y - \omega(1) > 0$ that is:*

$$(1-\gamma)(y-z) + \gamma\kappa > \frac{(1-s)}{3}\sqrt{\frac{2(1-s)\varphi t e^*}{\gamma}} > (1-\gamma)(1-s)t\left(\frac{1}{2} - \frac{(1-s)\gamma t}{12\varphi e^*}\right)$$

$\diamond$